

**Unified QCD evolution equations and the dominant behaviour
of structure functions at low x**

by

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ABSTRACT

We consider a system of evolution equations for quark and gluon structure functions satisfying the leading-logarithmic behaviour due to both QCD collinear (LLQ^2) and infrared ($LL1/x$) singularities. We show that these equations leave undetermined an arbitrary regular function of j in the Mellin-transformed weights. We consider the constraints resulting from energy-momentum conservation and from the decoupling of quark loops in the leading j -plane singularity. These constraints can be fulfilled without influencing the leading-log terms. As a particular consequence of the second constraint, the location of the leading singularity is determined in terms of the ($LL1/x$) and (LLQ^2) kernels. It leads to a value significantly lower than the $LL1/x$ evaluation, while remaining at $j > 1$, and compatible with the behaviour of structure functions observed at HERA.

The recent results on quark and gluon structure functions at Hera have paved the way for a reconsideration of QCD predictions. The observed behaviour of the proton structure function $F_2(x, Q^2)$ and some indications on the gluon structure function $F_G(x, Q^2)$ in the range $\{10^{-4} \leq x \leq 10^{-2}, 8 \leq Q^2 \leq 60 \text{ GeV}^2\}$ are characterized^[1] by a rapid rise at small x which is qualitatively compatible with the predictions of the resummation of leading $\log(1/x)$ contributions (hereafter denoted $LL1/x$) of the perturbative expansion, i.e. with the Lipatov singularity^[2] (BFKL). There exists quantitative studies of structure functions including the BFKL singularity^[3].

On the other hand, new tests of the celebrated Altarelli-Parisi evolution equations^[4] (DGLAP) are now possible in a much larger Q^2 range. These equations correspond to the resummation of the leading $\log(Q^2)$ terms (denoted LLQ^2) of the perturbative QCD expansion. Thus, HERA represents a unique apparatus for testing the QCD theoretical tools in kinematical domains where the perturbation expansion has to be resummed. As well-known, this is due to the appearance of collinear and infrared (in the infinite-momentum frame) singularities in the perturbative theory, implying large logarithms in the effective coupling constant. Hopefully, a precise comparison between experiment and theory will help understanding the yet unknown non-perturbative regime. In this context, a unified description of the $LL1/x$ and LLQ^2 evolution equations for quark and gluon structure functions is highly desirable.

Two different approaches have already been proposed, each one with its own advantages and inconveniences. First, it was remarked^[5] that Feynman diagrams for multi-gluon emission which contribute to $F_G(x, Q^2)$ are characterized by a common angular ordering property in the whole x -range, leading after resummation to the BFKL singularity at small x and to the DGLAP equations elsewhere. This allows one to write a unique equation for the whole x range^[6]. This approach leads to fruitful Monte Carlo simulations^[7] based on the multi-gluon diagrams but an explicit solution of the evolution equation^[6] itself has not yet been found.

In a second existing approach, explicit solutions have been proposed in a different context^[8]. Using a constructive procedure starting directly from the structure functions, a system of evolution equations can be written following the scheme:

$$(LL1/x) + (LLQ^2) - (DLL), \quad (1)$$

which indicates that the sum of $LL1/x$ and LLQ^2 contributions is corrected for double-counting by subtraction of the double-leading-logarithmic terms (DLL). This led the authors^[8] to propose a set of explicit equations for the quark-gluon system. However, as we shall see below, we find that the realization of the scheme (1) in terms of an explicit set of evolution equations leads to a solution which differs from that proposed in Ref.[8]. On the one hand, the set of equations obtained there leaves undetermined a regular function related to higher order contributions. On the other hand, the realization of (1) has to obey specific constraints which should be satisfied by the resulting equations, namely i) energy-momentum conservation, and ii) the decoupling of quark loops in the BFKL kernel. These constraints are not satisfied by the system of Ref.[8].

The goal of this paper is to provide an explicit realization of the formal scheme (1), obeying the constraints. We thus provide an explicit solution of the fundamental equation written in Ref.[6], while extending it to the quark-gluon system.

The generic set of equations we propose following the scheme (1) reads:

$$\begin{aligned} \frac{dF_G(j, Q^2)}{d \ln Q^2} &\equiv Q^2 f_G(j, Q^2) = Q_0^2 f_G(j, Q_0^2) + \int_{Q_0^2}^{Q^2} \frac{\alpha_S(Q'^2)}{4\pi} dQ'^2 \\ &\quad [\{(\nu_G + \Psi)(j) Q^2 K(Q^2, Q'^2) - \Psi(j)\} f_G(j, Q'^2) + \phi_F^G(j) f_S(j, Q'^2)] \\ \frac{dF_S(j, Q^2)}{d \ln Q^2} &\equiv Q^2 f_S(j, Q^2) = Q_0^2 f_S(j, Q_0^2) + \int_{Q_0^2}^{Q^2} \frac{\alpha_S(Q'^2)}{4\pi} dQ'^2 \\ &\quad [\nu_F(j) f_S(j, Q'^2) + 2n_F \phi_G^F(j) f_G(j, Q'^2)], \end{aligned} \quad (2)$$

where the inhomogeneous terms of the system of equations is implied by the compatibility with the DGLAP evolution equations near the threshold Q_0^2 . $\Psi(j)$ is an arbitrary function

provided it remains regular at $j = 1$, and where

$$\alpha_S(Q^2) = 1/(b \ln Q^2/\Lambda^2); \quad \{\nu_G, \phi_F^G, \nu_F, \phi_G^F\}(j),$$

are respectively the first-order QCD coupling constant, and the set of Altarelli-Parisi weights written with the usual conventions^[9]. Here $K(Q^2, Q'^2)$ is the BFKL kernel^[2]. Note that the specific equations of Ref.[7] are recovered with the choice $\Psi = 4N_c/(j-1) - \nu_G$. As we shall now demonstrate, this choice is not required by (1) and moreover it contradicts energy-momentum conservation and the decoupling of quark loops in the BFKL kernel.

Let us briefly show how the system (2) verifies both $LL1/x$ and LLQ^2 behaviour, while leaving undetermined the regular function $\Psi(j)$. It is well known^[8] that the kernel $K(Q^2, Q'^2)$ reduces simply to $1/Q^2$ when is retained only its contribution to the LLQ^2 terms. In that precise case, it is easy to realize by substitution in (2) that the function $\Psi(j)$ disappears from the system. Integrating by part Eqns.(2), one recovers the ordinary DGLAP evolution equations up to negligible terms.

The $LL1/x$ reduction of the system (2) is recovered by selecting the most singular contributions in the complex j -plane. This is obtained first by considering the equation with only the BFKL kernel $K(Q^2, Q'^2)$. Then one has also to select the most singular contribution from the prefactor $\nu_G + \Psi \approx 4N_c/(j-1)$. By this way, one finds back the usual BFKL kernel. Again, the function $\Psi(j)$ remains undetermined, provided that it is chosen regular at $j = 1$. By this discussion, we have proved that the requirement of equation (1) has a large class of solutions parametrized by the regular but arbitrary function $\Psi(j)$.

Now, let us introduce the constraints on the system of equations (2) which cannot be satisfied by the leading-log terms only. The first one is due to energy-momentum conservation which reads:

$$\forall Q^2, (F_G(j, Q^2) + F_S(j, Q^2))|_{j=2} = 1 \implies f_G(2, Q^2) + f_S(2, Q^2) = 0. \quad (3)$$

Inserting the constraint (3) (for both Q^2 and Q_0^2) in the system (2) taken at $j = 2$, one

gets:

$$0 \equiv \int_{Q_0^2}^{Q^2} \frac{\alpha_S(Q'^2) dQ'^2}{4\pi} (\Psi(2) + \nu_G(2)) (K(Q^2, Q'^2) - 1/Q^2), \quad (4)$$

where one uses the values of the Altarelli-Parisi kernels at $j = 2$, namely

$$\{\nu_G, \phi_F^G, \nu_F, \phi_G^F\}|_{j=2} \equiv \left\{ -\frac{2}{3}n_F, \frac{8}{3}C_2, -\frac{8}{3}C_2, \frac{1}{3} \right\}.$$

Then, if the kernel $K(Q^2, Q'^2)$ reduces to $1/Q^2$ the identity (4) is trivially fulfilled. Since it is not the case beyond the double-leading-log approximation, energy-momentum conservation requires the relation $\Psi(2) = \frac{2}{3}n_F$. This is the first condition on the previously undetermined function $\Psi(j)$.

Let us now consider solutions of the system (2). In the present paper, we shall restrict ourselves to the conventional case of a fixed coupling constant $\alpha_S \equiv \bar{\alpha}$. Indeed, this is the only case for which the BFKL singularity has been derived from the diagrammatic expansion^[2]. Following a common procedure, usual in the $LL1/x$ case^[2,8,10], we write the equations for $f_{G,S}$ in terms of their inverse Mellin transforms, namely:

$$f_{G,S}(j, Q^2) \equiv \int_C \frac{d\gamma}{2i\pi} \varphi_{G,S}(j, \gamma) (Q^2)^{\gamma-1}, \quad (5)$$

where C is a contour in the complex- γ plane leaving all singularities of the integrand at its left. Inserting the Mellin transform (5) in the system (2), and after some manipulations, one finds the following matrix equation:

$$\begin{pmatrix} 1 - \frac{\bar{\alpha}}{4\pi} \left\{ (\nu_G + \Psi) \omega(\gamma) - \frac{\Psi}{\gamma} \right\} & -\frac{\bar{\alpha}}{4\pi} \frac{\phi_F^G}{\gamma} \\ -\frac{\bar{\alpha}}{4\pi} \frac{2n_F \phi_G^F}{\gamma} & 1 - \frac{\bar{\alpha}}{4\pi} \frac{\nu_F}{\gamma} \end{pmatrix} \begin{pmatrix} \varphi_G \\ \varphi_S \end{pmatrix} = \begin{pmatrix} \varphi_{0G} \\ \varphi_{0S} \end{pmatrix} \quad (6)$$

where $\omega(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma)$, with $\psi(\gamma) = \frac{d \ln \Gamma}{d\gamma}$, is the well-known eigenvalue of the BFKL kernel¹. Note that the vector of the right-hand side (6) includes the boundary-

¹ The scale dependence of the system has been merged into the right-hand side of eq.(6), since it is not relevant for the computation of the BFKL singularity at fixed coupling. However a treatment with a running coupling constant would imply to take into account the threshold Q_0^2 , see for instance Ref.[10].

value conditions on the system (2). It is usually assumed not to contain the dominant j -plane singularity.

By matrix inversion one gets:

$$\begin{pmatrix} \varphi_G \\ \varphi_S \end{pmatrix} = \frac{1}{\mathcal{D}(j, \gamma)} \cdot \begin{pmatrix} 1 - \frac{\bar{\alpha}}{4\pi\gamma} \nu_F & \frac{\bar{\alpha}}{4\pi} \phi_F^G \\ \frac{\bar{\alpha}}{4\pi} 2n_F \phi_G^F & 1 - \frac{\bar{\alpha}}{4\pi} \left\{ \omega(\gamma) (\nu_G + \Psi) - \frac{\Psi}{\gamma} \right\} \end{pmatrix} \begin{pmatrix} \varphi_{0G} \\ \varphi_{0F} \end{pmatrix}, \quad (7)$$

where

$$\mathcal{D}(j, \gamma) = 1 - \frac{\bar{\alpha}}{4\pi} \left\{ \omega(\gamma) (\nu_G + \Psi) + \frac{1}{\gamma} (\nu_F - \Psi) \right\},$$

neglecting contributions of order $\bar{\alpha}^2$.

In order to obtain the leading behaviour of structure functions at low x , we perform a double inverse-Mellin transform on the system (7), first in the variable j , then in the variable γ :

$$f_{G,S}(x, Q^2) \equiv \int \frac{d\gamma}{2i\pi} e^{(\gamma-1)\ln Q^2} \int \frac{dj}{2i\pi} e^{(j-1)\ln 1/x} \varphi_{G,S}(j, \gamma). \quad (8)$$

The first integral of Eq. (8) (in the complex j -plane) is given by the residue of the pole at $j = j_p(\gamma)$ where $\mathcal{D}(j_p, \gamma) = 0$. Then the second Mellin transform (in the complex γ -plane) is performed by a saddle-point method. In the domain of validity of the BFKL singularity, one assumes $\frac{\ln Q^2}{\ln 1/x} \ll 1$, and the saddle-point is at $\gamma = \gamma_c$ corresponding to $\frac{dj_p(\gamma_c)}{d\gamma} = 0$. One finds:

$$\begin{aligned} (pole) \quad \frac{4\pi}{\bar{\alpha}} &= (\nu_G(j_p) + \Psi(j_p)) \omega(\gamma) + (\nu_F(j_p) - \Psi(j_p)) \frac{1}{\gamma} \\ (saddle point) \quad 0 &= (\nu_G(\bar{j}) + \Psi(\bar{j})) \omega'(\gamma_c) - \frac{1}{\gamma_c^2} (\nu_F(\bar{j}) - \Psi(\bar{j})), \end{aligned} \quad (9)$$

where $\bar{j} = j_p(\gamma_c)$. The resulting structure functions behave like:

$$f_{G,S}(x, Q^2) \approx [Q^2]^{\gamma_c-1} x^{-(\bar{j}-1)}, \quad (10)$$

Now, we are able to express the second constraint on the system of equations (2), namely the decoupling of quark loops from the BFKL singularity kernel. Let us first

require that the dominant behaviour (10) agrees with the theoretical determination of the BFKL singularity^[2,8]. It corresponds to the well-known “critical” value of γ , namely $\gamma_c = 1/2$, with $\omega'(\gamma_c) = 0$ and $\omega(\gamma_c) = 4 \ln 2$. From inspection of the system (9), this implies the following relations:

$$\begin{aligned}\Psi(\bar{j}) &= \nu_F(\bar{j}) \\ \nu_G(\bar{j}) + \nu_F(\bar{j}) &= \left[\frac{4\bar{\alpha} \log 2}{4\pi} \right]^{-1}.\end{aligned}\tag{11}$$

We note that the first of relations (11) implies the cancellation of the quark contribution $\nu_F - \Psi$ in the pole location, see (7). Indeed, it is known^[2] that a loop contribution for a particle of spin s is of order $x^{-(2s-1)}$ at small x . Thus, quark loops are negligible in the BFKL kernel.

The results of Eqs.(11) have some interesting consequences on the location of the BFKL singularity in the j -plane.

If we retain only the dominant behaviour $\nu_G \approx \frac{4N}{j-1}$, when $j \mapsto 1$, we recover the usual determination of the BFKL singularity, in the absence of LLQ^2 contributions. The coupling to the LLQ^2 terms, which is allowed by our unified description, is responsible for the modified formula (11). As an illustration, let us for instance fix the value of $\bar{\alpha}$ such that $\frac{N_c}{\pi} \bar{\alpha} 4 \ln 2 \simeq .5$. This value is often considered as fixing the “bare” BFKL singularity^[3]. Using this value and fixing $N_c = n_F = 3$, one gets from the second equation (11): $\bar{j} - 1 = 0,313$. Quite interestingly, this value is in better agreement with those phenomenologically obtained from Hera experimental results, if analyzed directly in terms of a BFKL singularity^[1]. Note that, with these input numbers, one gets for the function Ψ the two constraints $\Psi(\bar{j}) \equiv \nu_F(j = 1.313) \simeq -1.44$; $\Psi(2) \equiv \nu_G(2) = 2$. The kinematical constraint of energy-momentum conservation decreases the intercept of the BFKL singularity^[11], through non-leading terms in the $LL1/x$ expansion. However, this constraint at $j = 2$ does not allow a determination of the modified location of the singularity. In our derivation, see equation (11), we obtain a precise theoretical prediction for this

shift, based on a different constraint, e.g. the decoupling of the quark loops from the BFKL kernel.

Note that in all realistic cases (values of n_F , flavour thresholds, etc...), $\nu_G(j) + \nu_F(j) < 4N/(j-1)$, for $j > 1$. Thus the decoupling constraint always decreases the BFKL intercept $\bar{j} - 1$.

In conclusion, we have proposed QCD evolution equations combining the leading-logarithmic contributions in both the x and Q^2 variables and satisfying constraints, namely energy-momentum conservation and quark decoupling from the BFKL singularity. Our results extend the QCD predictions beyond the leading-logarithm contributions. In particular, we find the location of the dominant singularity of structure functions at small- x and fixed coupling constant, which is shifted down with respect to the $LL1/x$ prediction. Other consequences of this improved leading-logarithm scheme should be explored. In particular, it would be interesting to see in which way the system (2), with an adequate function $\Psi(j)$, gives an explicit realization of the gluon resummation of Ref.[5].

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